

Large order asymptotics of semiclassical expansion: a new approach

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Abstract

A new approach to the problem of finding the asymptotical behaviour of large orders of semiclassical expansion is suggested. Asymptotics of high orders not only for eigenvalues, but also for eigenfunctions, are constructed. Thus, one can apply not only functional integral technique, which has been used up to now, but also method of direct analysis of the semiclassical expansion recursive relations.

1 Introduction

Problems associated with large order behaviour of perturbation series coefficients were examined in many papers. It was F.Dyson [1] who argued in 1952 that perturbation series in quantum field theory diverges. Asymptotics of coefficients at high orders which is useful, for example, for investigation of the divergent series summation problem was found later.

This asymptotics is usually constructed by the technique used by L.Lipatov [2] in quantum field theory and by E.Brézin, J.C.Le Guillou and J.Zinn-Justin [3] in quantum mechanics. This method is the following. The k -th order of perturbation theory for quantities like Green functions can be represented through a functional integral which can be approximately calculated by saddle-point technique. This approach was reusable in quantum mechanics (see, for example, [3, 4, 5, 6]) for study of the high orders of perturbation theory for ground state energy and the n -th excited state energy as n is not large.

In this paper a new method for constructing high order asymptotics is suggested. This technique allows us to find such asymptotics not only for eigenvalues but also for eigenfunctions. This method is based not on the functional integral approach but on direct analysis of recursive relations.

Consider the following dependence of the Hamiltonian on a perturbation theory parameter g , momenta $p = (p_1, \dots, p_n)$ and coordinates $x = (x_1, \dots, x_n)$:

$$\mathcal{H} = \frac{p^2}{2} + \frac{1}{g^2}V(gx), \quad (1)$$

where $p_m = -i\partial/\partial x_m$, the potential V has a local minimum at $x = 0$. These Hamiltonians were considered in [3, 4, 5]. Without loss of generality, one can assume that $V(Q) \sim Q^2/2 + O(Q^3)$ as $Q \rightarrow 0$.

Contrary to the perturbation expansion in powers of g for eigenvalues, perturbation expansion for eigenfunctions can be constructed in different ways. For in-

stance, the wave function Ψ can be simply expanded in powers of g at fixed x , $\Psi(x) = \sum_{k=0}^{\infty} g^k \Psi_k(x)$. But on the other hand, one can first carry out the following change of the wave function argument,

$$gx = Q, \quad (2)$$

and expand the wave function at fixed Q . It appears that in this case the expansion in powers of g^2 is a tunnel semiclassical expansion (see, for example, [7, 8]). The square of perturbation theory parameter becomes the Planck constant analog, because the equation for eigenfunction $\mathcal{H}\Psi = E_0\Psi$ after multiplying it through by g^2 , substitution (2) and renotation $g^2 = \hbar$ takes the form

$$-\frac{\hbar^2}{2}\Delta\Psi + V(Q)\Psi = \hbar E_0\Psi. \quad (3)$$

As it is known (see, for example, [7, 8]), the tunnel asymptotics of the ground state wave function has the form of a product of a slowly varying pre-exponential factor by a rapidly varying exponential function

$$\Phi(Q, \hbar) \exp\left(-\frac{1}{\hbar}S(Q)\right). \quad (4)$$

Substitution of this formula to eq.(3) gives the Hamilton-Jacobi equation for function S

$$(\nabla S(Q))^2/2 = V(Q) \quad (5)$$

and the following equation for function Φ ,

$$-\frac{\hbar}{2}\Delta\Phi + \nabla S\nabla\Phi + \frac{1}{2}(\Delta S - n)\Phi = (E_0 - n/2)\Phi. \quad (6)$$

One can apply perturbation theory to it. Both function Φ and ground state energy can be expanded in powers of \hbar ,

$$\Phi(Q, \hbar) = \sum_{k=0}^{\infty} \hbar^k \Phi_k(Q), E_0(\hbar) = \sum_{k=0}^{\infty} \hbar^k E^{(k)}, E^{(0)} = n/2.$$

This paper deals with the asymptotical behaviour of $\Phi_k(Q)$ as k is large and Q is fixed. The problem of the high order asymptotics of the eigenfunction perturbation theory when the expansion is considered at fixed x will be discussed in the next paper.

In one-dimensional quantum mechanics of a particle in a potential shown in fig.1 which was considered in [3, 5] the exponent in (4) is expressed through the action S on the trajectory starting from zero and reaching at once the point Q (solid line in fig.2). This trajectory is the solution of the euclidean equation of motion which can be obtained from the ordinary one by changing of the real time t to the imaginary one $-i\tau$:

$$\frac{d^2}{d\tau^2} Q = V'(Q). \quad (7)$$

It appears that the asymptotics of the quantity Φ_k at larges k can be expressed through the action S_s on the other euclidean solution which starts from zero, reaches the turning point and finishes also at point Q (dashed line in fig.2).

As well as in one-dimensional case, in multidimensional case the main contribution to the wave function is given by the classical euclidean solution with the least action S , while high order asymptotics of the semiclassical expansion is determined by the action S_s on another euclidean solution.

It is shown in this paper that high orders of Φ_k have the following asymptotic behaviour at larges k ,

$$\Phi_k(Q) \sim \frac{(k-1)!}{(S_s(Q) - S(Q))^k}, \quad (8)$$

the pre-exponential factor is omitted in this formula. From the expressions obtained in this paper one can also find the asymptotic behaviour of the ground state energy perturbation coefficients $E^{(k)}$ and carry out a check of the formulas obtained in [3, 4, 6] for various cases by the path integral technique.

Besides the potential shown in fig.1, other examples are also considered in this paper. First, the case of a particle moving in a radial symmetric potential in n -

dimensional space is also considered. If the potential depends on the distance of the origin as it is shown in fig.1, then classical trajectories determining the asymptotics of the eigenfunction and the high order behaviour of semiclassical expansion are analogous with the trajectories shown in fig.2 by solid and dashed lines for one-dimensional case.

Second, the example of the potential with degenerate minima (fig.3) is also discussed. Classical solution determining the asymptotics of the eigenfunction is analogous to shown in fig.2. On the other hand, not classical solution but "almost classical solution" contributes to the large order behaviour of semiclassical expansion. This "almost solution" starts at $\tau \rightarrow -\infty$ from the origin, then transits to another minimum. The "almost solution" resides in this minimum for a long euclidean time and finally reaches the point Q . As it was shown in [4, 5], if $Q = 0$, then these "almost solutions" (instanton-anti-instanton pairs) contribute to large order behaviour of perturbation theory for the ground state energy.

Finally, a particle on n -dimensional sphere in the external potential depending only on one coordinate and having one minimum at the south pole of the sphere which is a classical ground state (fig.4) is also considered in this paper. One of the classical solutions shown in fig.4 by solid line determines the behaviour of eigenfunction, another solution passing through the north pole of the sphere determines the high order asymptotics of semiclassical expansion. This solution is shown in fig.4 by dashed line. An interesting feature of this asymptotics is the nullification of it at odd n which was found in ref.[6] for the ground state energy perturbation theory. In this case high order asymptotics is determined by the "almost solution" which makes a loop around the sphere, resides in the south pole for a long time and finally reaches the point Q .

2 Methods of finding the large order asymptotics of semiclassical expansion

Asymptotics of Φ_k can be found by various methods. First, the ground state wave function can be expressed through the path integral (see, for example, [9]) over trajectories starting as $\tau \rightarrow -\infty$ from zero and reaching the point Q at $\tau = 0$:

$$\int_{\mathcal{Q}(-\infty)=0, \mathcal{Q}(0)=Q} D\mathcal{Q} \exp\left(-\frac{1}{\hbar} S[\mathcal{Q}]\right) \quad (9)$$

where S is the euclidean action of the theory and have the form:

$$S[\mathcal{Q}] = \int d\tau [\dot{\mathcal{Q}}^2/2 + V(\mathcal{Q})]. \quad (10)$$

This integral can be evaluated as $\hbar \rightarrow 0$ by the Laplace method. As an exponential approximation this integral is equal to

$$\exp\left(-\frac{1}{\hbar} \min_{\mathcal{Q}(-\infty)=0, \mathcal{Q}(0)=Q} S[\mathcal{Q}]\right), \quad (11)$$

This formula coincides with semiclassical tunnel asymptotics. The pre-exponential factor and the corrections can be calculated with the help of extraction of the factor (11) from the path integral (9), substitution $\mathcal{Q} = \mathcal{Q}_0 + q\sqrt{\hbar}$, where \mathcal{Q}_0 is the trajectory with the least action in eq.(11), expansion of the integrand after this substitution in terms of $\sqrt{\hbar}$ and calculation of integrals of the products of some polynomial in q functions by the Gauss exponent. Coefficients of odd powers in $\sqrt{\hbar}$ are equal to zero. Quantities Φ_k can be expressed through the integrals

$$\Phi_k = \int Dq \oint_C \frac{dg}{2\pi i g^{2k+1}} \exp\left(-\frac{1}{g^2} [S(\mathcal{Q}_0 + gq) - S(\mathcal{Q}_0)]\right), \quad (12)$$

where the contour C dependend in general on q runs around the origin counterclockwise. After substitution $\mathcal{Q}_0 + gq = \mathcal{Q}$, $g = \nu/\sqrt{k}$ integrals (12) take the form:

$$\Phi_k = k^k \int \frac{D\mathcal{Q} d\nu}{2\pi i \nu^{2k+1}} \exp\left(-k \left(\frac{S[\mathcal{Q}] - S[\mathcal{Q}_0]}{\nu^2} + \ln \nu^2\right)\right), \quad (13)$$

and can be evaluated by saddle-point technique.

Consider saddle points of the exponent (\mathcal{Q}_s, ν_s) . It's variation in ν gives us the following condition, $\nu_s^2 = S(\mathcal{Q}_s) - S(\mathcal{Q}_0)$, variation in Φ leads to the condition $\delta S(\mathcal{Q}_s) = 0$, i.e. \mathcal{Q}_s is the classical solution. Therefore, asymptotics of the integral (13) has the form (8) up to a pre-exponential factor, where $S_s(Q) = S[\mathcal{Q}_s]$, $S(Q) = S[\mathcal{Q}_0]$.

Another technique to calculate the Φ_k high order asymptotics is based on the direct analysis of the recursive relations for the Φ_k which can be obtained from eq.(6)

$$-\frac{1}{2}\Delta\Phi_{k-1} + \nabla S \nabla \Phi_k + \frac{1}{2}(\Delta S - n)\Phi_k = \sum_{p=1}^k E^{(p)}\Phi_{k-p} \quad (14)$$

Let us look for the large order asymptotics in a form

$$\Phi_k \sim \frac{(k-1)!}{A(Q)^k}. \quad (15)$$

Substitution of formula (15) to relations (14) gives the following equation for A

$$\nabla A = -2\nabla S \quad (16)$$

in the leading order. It has been supposed that high orders of Φ_k are growing faster than $E^{(k)}$. This assumption can be justified as follows. It follows from eq.(16) that the value of the function A in zero is more than in other points of some vicinity of zero. Since the $E^{(k)}$ asymptotics is distinguished from the Φ_k asymptotics only by a pre-exponential factor and, therefore, has the form $\Gamma(k)/A(0)^k$, the right-hand side of eq.(14) can be neglected at $Q \neq 0$ because it is exponentially small. On the other hand, when $A(0) - A(Q) \sim 1/k$, one can't ignore the right-hand side, so that calculations based on this neglect break down at $Q \sim 1/\sqrt{k}$. Therefore, the asymptotics at these Q and larges k has another form discussed in section 5.

Section 3 contains the calculation of the pre-exponential factor in formula (15) with the help of direct analysis of the recursive relations. This factor is defined up to a multiplier, the function A is determined up to an additive constant.

In section 3 there is also a consideration of the following interesting method for calculating the pre-exponential factor and the corrections to asymptotic formula. Let us look for the Φ_k asymptotics as $k \rightarrow \infty$ when the corrections are taken into account in a form

$$\Phi_k = B(Q, e^{-\partial/\partial k}) \frac{(k-1)!}{A(Q)^k} \quad (17)$$

where B is a sum of power functions of the operator $e^{-\partial/\partial k}$. Notice that the operator $e^{-\partial/\partial k}$ plays the role of a small parameter in this case. Namely, it transforms the sequence $(k-1)!/A^k$ to the sequence $(k-2)!/A^{k-1}$, the k -th order of the latter sequence is less than the k -th order of the former one approximately in k/A times. Therefore, any asymptotics of the form

$$k^\nu \frac{(k-1)!}{A(Q)^k} (1 + a_1/k + a_2/k^2 + \dots) \quad (18)$$

can be presented in a form

$$(b_0 + b_1 e^{-\partial/\partial k} + b_2 e^{-2\partial/\partial k} + \dots) e^{\nu\partial/\partial k} \frac{(k-1)!}{A(Q)^k} \quad (19)$$

i.e. in the form (17). Coefficients b in eq.(19) can be expressed through the coefficients a in eq.(18). The k -th order of the ground state energy perturbation theory can be presented in the form analogous with (17), too.

These expansions for $E^{(k)}$ and Φ_k can be substituted to the relations (14). Analysis of the obtained equation is analogous to calculating of the corrections to semiclassical approximation. But in this case the parameter of the expansion is not a number (Planck constant) but an operator $e^{-\partial/\partial k}$. The general theory of the semiclassical expansion when \hbar is an operator was developed in [8].

Eq.(16) determines the function A up to an additive constant. For determining it, one must analyse the behaviour of semiclassical approximation near singular points: in one-dimensional case it is the turning point that gives the singularity in eq. (8).

The behaviour of semiclassical expansion near the turning point which determines the constants in formulas for the function A and the pre-exponential factor is analysed in section 4 for various types of turning points.

It will be shown that the pre-exponential factor is divergent near the point $Q = 0$. The divergence is connected with the necessity of constructing another asymptotics as $Q \sim 1/\sqrt{k}$, which is non-singular and allows us to find large order asymptotics of the ground state energy perturbation theory. Section 5 contains such derivation. The obtained results coincide with the obtained one in refs.[3, 4, 6].

3 Analysis of the semiclassical expansion recursive relations

In this section the derivation of the asymptotic formulas from the recursive relations is considered in more details. Let us examine the following form for the asymptotics of Φ_k at larges k

$$\Phi_k(Q) \sim \frac{\Gamma(k)k^\nu}{A(Q)^{k+\nu}} f(Q) \quad (20)$$

and find conditions for the constant ν and functions A , f . Substitute the expression (20) to the left-hand side of eq.(14) and consider first terms of order $\Gamma(k)k^{\nu+1}$ and then terms of order $\Gamma(k)k^\nu$. To calculate the corrections to the asymptotics (20) one can consider the following terms.

Let us use the relations

$$\nabla \Phi_k = -\frac{\Gamma(k)k^\nu(k+\nu)}{A(Q)^{k+\nu+1}} f(Q) \nabla A + \frac{\Gamma(k)k^\nu}{A(Q)^{k+\nu}} \nabla f + \dots, \quad (21)$$

$$\begin{aligned} \Delta \Phi_{k-1} &= \frac{\Gamma(k)k^\nu(k+\nu)}{A(Q)^{k+\nu+1}} (\nabla A)^2 f(Q) \\ &+ \frac{\Gamma(k)k^\nu}{A(Q)^{k+\nu}} [-f(Q) \Delta A - 2 \nabla A \nabla f] + \dots \end{aligned} \quad (22)$$

The right-hand side of eq.(14) can be omitted if $Q \neq 0$ because of the remark of section 2. Substitution of the relations (21) and (22) to formula (14) gives the equation for A (16) in a leading order, so that

$$A(Q) = A_0 - 2S(Q), A_0 = \text{const.}$$

The next order gives us the following equation for f ,

$$\frac{1}{2}[f\Delta A + 2\nabla A\nabla f] + \nabla S\nabla f + \frac{1}{2}(\Delta S - n)f = 0. \quad (23)$$

Separate now factor Φ_0 from the function f and denote

$$f/\Phi_0 = X. \quad (24)$$

It follows from the equations for Φ_0 and A that the function $X(Q)$ satisfies the condition,

$$\nabla S\nabla X = -nX \quad (25)$$

and can be denote up to a multiplier. In one-dimensional case function X has the form

$$X = c \exp\left(\int_Q^{Q^+} dQ/\sqrt{2V(Q)}\right) \quad (26)$$

In section 4 the vicinity of the turning point is considered in more details and constants c and ν are found. Other types of singular points are also considered.

Recursive relations for semiclassical expansion can be also investigated for Hamiltonians with quantum corrections. For example, consider Hamiltonians of the following type [6],

$$H = -\frac{\hbar^2}{2} \frac{d^2}{dQ^2} - \frac{n-1}{2Q} u(Q) \hbar^2 \frac{d}{dQ} + V(Q), \quad (27)$$

where function $u(Q)$ is equal to 1 when $Q = 0$. When $u = Q \text{ctg} Q$, this Hamiltonian describes a particle on n -dimensional sphere. If $u = 1$ then eq.(27) corresponds to the $O(n)$ -symmetrical case.

Substitution of the formula (4) to the equation $H\psi = \hbar E_0\psi$ leads to the Hamilton-Jacobi equation coinciding to (5), while the equation for Φ takes the more complicated form than (6):

$$-\frac{\hbar}{2}\Phi'' + S'\Phi' + \frac{1}{2}(S'' - n)\Phi + \frac{n-1}{2Q}uS'\Phi - \hbar\frac{n-1}{2Q}u\Phi' = (E_0 - n/2)\Phi.$$

Recursive relations for the perturbation series coefficients in \hbar can be also obtained from the equation for Φ .

Asymptotics of the Φ_k at larges k can be also looked for in a form (20). Substitution of this formula to the recursive relations gives the equation for f distinguished from (23). But after extracting the factor Φ_0 (eq.(24)) the relation for the function X takes, nevertheless, the form (25). Its solution has the form

$$X = c \exp \left(n \int_Q^{Q_+} \frac{dQ}{\sqrt{2V(Q)}} \right).$$

One can find the corrections to the asymptotic formula (20) by evaluating the following terms in eqs.(21), (22), substituting it to the left-hand side of the eq. (14) and setting the coefficients of the corresponding orders in k equal to zero. Nevertheless, in this section another technique to find the corrections is examined. This method illustrates the analogy between the corrections to semiclassical approximation and the corrections to high order asymptotics.

As it has been mentioned in section 2, let us search for the asymptotics of the Φ_k in a form (17). When $Q \neq 0$, one can present the eq.(14) in a form

$$\begin{aligned} -\frac{1}{2}e^{-\partial/\partial k}\Delta\Phi_k + \nabla S\nabla\Phi_k + \frac{1}{2}(\Delta S - n)\Phi_k = \\ = (E^{(1)}e^{-\partial/\partial k} + E^{(2)}e^{-2\partial/\partial k} + \dots)\Phi_k, \end{aligned} \quad (28)$$

where the exponentially small terms associated with the ground state energy asymptotics are omitted.

As it has been shown in section 2, the operator $e^{-\partial/\partial k}$ plays a role of a small parameter analogous to \hbar in a case of semiclassical expansion.

Notice that the commutation rule of the derivation operator and the function $\Gamma(k)A(Q)^{-k}$ can be presented in a form

$$\frac{\partial}{\partial Q} \frac{\Gamma(k)}{A(Q)^k} = -e^{-\partial/\partial k} \frac{\Gamma(k)}{A(Q)^k} \frac{\partial A}{\partial Q} + \frac{\Gamma(k)}{A(Q)^k} \frac{\partial}{\partial Q} \quad (29)$$

analogous to the commutation formula of the derivation operator with the exponent

$$\frac{\partial}{\partial Q} \exp(-A/\hbar) = -\frac{1}{\hbar} \exp(-A/\hbar) \frac{\partial A}{\partial Q} + \exp(-A/\hbar) \frac{\partial}{\partial Q},$$

where a number \hbar is substituted by the operator $e^{-\partial/\partial k}$, while the function of \hbar , $\exp(-A/\hbar)$, is changed by the set of numbers, $\Gamma(k)A^{-k}$. Application of eq.(29) to eq.(28) leads to the equation for the function B presented in formula (17),

$$\begin{aligned} & -\frac{e^{-\partial/\partial k}}{2}(\nabla - e^{\partial/\partial k}\nabla A)^2 B + \nabla S(\nabla - e^{\partial/\partial k}\nabla A)B \\ & + \frac{1}{2}(\Delta S - n)B = (e^{-\partial/\partial k}E^{(1)} + e^{-2\partial/\partial k}E^{(2)} + \dots)B \end{aligned}$$

which is analogous to the equation for Φ (6). Let us set the coefficients of each order in $e^{-\partial/\partial k}$ equal to zero. We obtain the equations derived earlier and the following formula for the corrections,

$$\begin{aligned} B(e^{-\partial/\partial k}, Q) &= B_0(Q) \sum_{l=0}^{\infty} F_l(Q) e^{-l\partial/\partial k}, \\ -\frac{1}{2B_0} \Delta(F_{l-1}B_0) + \nabla S \nabla F_l &= \sum_{s=0}^{l-1} E^{(l-s)} F_s \end{aligned}$$

These equations define the function B up to the factor which does not depend on Q but depends on $e^{-\partial/\partial k}$. Let us consider the singular points and find this factor.

4 The behaviour of the asymptotics near singular points

As it has been noticed in the previous section, one must consider the behaviour of the recursive relations near singular points in order to find constants c, A_0, ν . This problem is examined in this section. The main idea of the consideration is the following. The potential near singular point can be approximated by a linear, quadratic or another function depending on the type of a singular point. All the orders of semi-classical expansion can be evaluated exactly for this approximate potential, so one can find unknown constants by comparing the "exact" results with the asymptotics discussed in section 3.

In this section three types of one-dimensional singularities are discussed:

- i) ordinary turning point (fig.1);
- ii) potentials with degenerate minima (fig.3);
- iii) singular quantum correction to the Hamiltonian [6].

Consider these cases more precisely.

4.1 Ordinary turning point

In this subsection the case of one-dimensional quantum mechanics of a particle in the potential shown in fig.1 is considered. Let us discuss the behaviour of semiclassical expansion for the ground state wave function in the vicinity of point Q_+ . The potential V can be approximated by a linear function,

$$V \sim a\xi, \xi = Q_+ - Q, \quad (30)$$

The function A satisfying the equation (16) is approximately equal to

$$A = A_0 - 2S_+ + \frac{4}{3}\sqrt{2a}\xi^{3/2},$$

where S_+ is the action $S = \int_0^{Q_+} \sqrt{2V(Q)} dQ$ at the turning point Q_+ . As the duration of motion from the point Q to the turning point has the form $\int_Q^{Q_+} \frac{dQ}{\sqrt{2V(Q)}} = \sqrt{2\xi/a}$, eq.(20) implies that the high orders of semiclassical expansion behave as follows,

$$\Phi_k \sim \frac{\Gamma(k) k^\nu c e^{\sqrt{2\xi/a}}}{(A_0 - 2S_+ + \frac{4}{3}\sqrt{2a\xi}^{3/2})^{k+\nu}} \Phi_0. \quad (31)$$

Let us obtain the asymptotics (31) in another way and find the coefficients c, ν, A_0 .

First of all, notice that the semiclassical expansion for the wave function in the case of a potential (30) can be obtained as follows. Formula (28) implies that the function $\psi = \sum \Phi_k \hbar^k e^{-S/\hbar}$ approximately satisfies as an asymptotic series the equation

$$\left(-\frac{\hbar^2}{2} \frac{d^2}{d\xi^2} + a\xi\right) \psi(\xi) = 0$$

The expansion in powers of \hbar of the growing at large ξ solution can be obtained from the following expression,

$$\psi(\xi) = \int dp \exp\left(-\frac{1}{\hbar}\left(p\xi - \frac{p^3}{6a}\right)\right) \quad (32)$$

where the integral is taken over any sufficiently small region containing the minimum of the exponent, $p_0 = -\sqrt{2a\xi}$. It follows from the integral presentation (32) that the Φ_k asymptotics at large k has the form

$$\Phi_k \sim \Phi_0 \frac{1}{2\pi} \frac{\Gamma(k) e^{\sqrt{2\xi/a}}}{\left(\frac{4}{3}\xi\sqrt{2a\xi}\right)^k},$$

coinciding with eq.(31) when

$$\nu = 0, A_0 = 2S_+, c = 1/(2\pi). \quad (33)$$

Substitution of these constants to the formulas obtained in section 3 gives us the following asymptotics,

$$\Phi_k(Q) \sim \frac{(k-1)! \Phi_0(Q)}{2\pi (S_s(Q) - S(Q))^k} \exp\left(\int_Q^{Q_+} \frac{dQ}{\sqrt{2V(Q)}}\right). \quad (34)$$

4.2 Potential with degenerate minima

Consider now the case of the double-well potential shown in fig.3. Namely, let the potential V have the minimum in the point Q_+ , besides the minimum at zero and let $V(Q_+)$ equal to zero. In the vicinity of the point Q_+ the potential can be approximated by a quadratic function,

$$V \sim \omega^2 \xi^2 / 2, \xi = Q_+ - Q.$$

Action S has the following form in this approximation,

$$S = S_+ - \omega \xi^2 / 2$$

The duration of motion from point Q to point Q_+ is infinite in this case, contrary to the previous subsection. Therefore, the solution X to eq. (25) has the more complicated form than (26). Namely,

$$X = c(Q_+ - Q)^{1/\omega} \exp \int_Q^{Q_+} dQ \left[\frac{1}{\sqrt{2V(Q)}} - \frac{1}{\omega(Q_+ - Q)} \right], \quad (35)$$

in this formula singular contribution is subtracted from the exponent in eq.(26) and the normalizing factor is redefined. One can show by the explicit calculation that the function (35) really satisfies eq.(25).

In the vicinity of the point Q_+ the asymptotic formula (20) takes the form

$$\Phi_k \sim \frac{\Gamma(k) k^\nu c \xi^{1/\omega}}{(A_0 - 2S_+ + \omega \xi^2)^{k+\nu}} \Phi_0 \quad (36)$$

On the other hand, recursive relations have the following form in the quadratic approximation

$$-\frac{1}{2} \frac{d^2 \Phi_{k-1}}{d\xi^2} - \omega \xi \Phi_0 \left(\frac{\Phi_k}{\Phi_0} \right)' = 0 \quad (37)$$

and can be solved exactly,

$$\Phi_k = \frac{c_k}{\xi^{2k}} \Phi_0, \Phi_0 = \xi^{-\frac{1+\omega}{2\omega}} \quad (38)$$

where numerical coefficients c_k have the form

$$c_k = \frac{\Gamma(2k + \frac{1+\omega}{2\omega})}{(4\omega)^k \Gamma(k+1) \Gamma(\frac{1+\omega}{2\omega})}$$

Making use of the Stirling formula for the Gamma-function, one can obtain that at large k the c_k asymptotics can be written in a form

$$c_k \sim \frac{(k-1)! (2k)^{1/\omega}}{\omega^k \sqrt{2\pi} \Gamma(\frac{1+\omega}{2\omega})} \quad (39)$$

The constants in the Φ_k asymptotics can be found by comparing the formulas (39),(38) with the formula (36). The constants are the following,

$$c = \frac{(2\omega)^{\frac{1}{2\omega}}}{\sqrt{2\pi} \Gamma(\frac{1+\omega}{2\omega})}, \nu = \frac{1}{2\omega}, A_0 = 2S_+. \quad (40)$$

It follows from the formulas (20),(24), (40),(35) that the k -th order of the ground state wave function semiclassical expansion has the following asymptotics as $k \rightarrow \infty$ in the case of a degenerate minima potential,

$$\begin{aligned} \Phi_k \sim \Phi_0 & \frac{\Gamma(k) k^{1/2\omega}}{(2S_+ - 2S(Q))^{k+1/2\omega}} \frac{(2\omega)^{\frac{1}{2\omega}} (Q_+ - Q)^{1/\omega}}{\sqrt{2\pi} \Gamma(\frac{1+\omega}{2\omega})} \\ & \times \exp \int_Q^{Q_+} \left[\frac{1}{\sqrt{2V(Q)}} - \frac{1}{\omega(Q_+ - Q)} \right] \end{aligned} \quad (41)$$

4.3 Singular quantum correction to the Hamiltonian

Consider the Hamiltonians presented as a sum of a classical (non-singular) Hamiltonian and a quantum correction to it which is singular,

$$H = -\frac{\hbar^2}{2} \frac{d^2}{dQ^2} + V(\cos Q) - \frac{\hbar^2}{2} (n-1) \frac{\cos Q}{\sin Q} \frac{d}{dQ} \quad (42)$$

Evaluation of the high order asymptotics for the ground state energy perturbation theory in this case was considered in [6]. The Φ_k asymptotics in this case has the form (20), the function X satisfies the equation (25) at non-singular points ($Q \in (0, \pi)$) and, therefore, has the form (26), where $Q_+ = \pi$.

When one study the vicinity of the singular point in this case, one must take into account not only classical part of the Hamiltonian but also the quantum correction to it because it is singular, as oppose to the previous case. The classical Hamiltonian can be approximated by the Hamiltonian of a free particle,

$$-\frac{\hbar^2}{2} \frac{d^2}{d\xi^2} + E, \xi = \pi - Q, \quad (43)$$

while the quantum correction is approximately equal to the following operator,

$$-\frac{\hbar^2}{2}(n-1)\frac{1}{\xi}\frac{d}{d\xi}. \quad (44)$$

In the vicinity of the singular point the asymptotic formula (20) takes the following approximate form,

$$\Phi_k \sim \frac{\Gamma(k)k^\nu c e^{\xi/\sqrt{2E}}}{(A_0 - 2S_+ + 2\sqrt{2E}\xi)^{k+\nu}} \Phi_0 \quad (45)$$

The constants can be found by comparing with this formula. The function $\psi = \sum \hbar^k \Phi_k e^{-S/\hbar}$ satisfies the equation

$$\left(-\frac{\hbar^2}{2} \frac{d^2}{d\xi^2} + E - \frac{\hbar^2}{2}(n-1)\frac{1}{\xi}\frac{d}{d\xi}\right)\psi(\xi) = 0.$$

Its solution growing at infinity is considered. It can be expressed through the Infeld function $I_{n/2-1}$ (the Bessel function of a purely imaginary argument),

$$\psi(\xi) = \xi^{1-n/2} I_{n/2-1}[\xi\sqrt{2E}/\hbar].$$

Semiclassical expansion of the fnction ψ corresponds to the Infeld function expansion at large arguments. As this asymptotic series breaks for the Infeld function of the half-integer order, semiclassical expansion also breaks, so in the approximation (43),(44) all the orders of Φ_k begining from some k are equal to zero. Therefore, singular point $Q_+ = \pi$ does not contribute to the high order asymptotics of semiclassical expansion at integer odd n . This confirms the assumption of [6].

If n is not equal to an odd integer number, the Infeld function expansion coefficients

$$I_\nu(x) = \frac{1}{\sqrt{2\pi x}} e^x (1 + c_1/x + c_2/x^2 + \dots)$$

have the form [10]:

$$c_k = \frac{\cos \pi \nu}{\pi} \frac{\Gamma(k - \nu + 1/2) \Gamma(k + \nu + 1/2)}{2^k k!}.$$

and the following asymptotic behaviour at larges k

$$c_k \sim \frac{\Gamma(k)}{2^k} \frac{\cos \pi \nu}{\pi}.$$

Therefore, the semiclassical expansion of the function ψ behaves at high orders as follows,

$$\Phi_k \sim \Phi_0 \frac{\Gamma(k)}{(2\xi)^k (2E)^{k/2}} \frac{\cos \pi(n/2 - 1)}{\pi}$$

Therefore, the constants A_0, c, ν in the asymptotic formula for Φ_k have the form,

$$A_0 = 2S_+, \nu = 0, c = -\frac{\cos(\pi n/2)}{\pi}$$

Thus, the following expression for the Φ_k asymptotics is obtained,

$$\Phi_k \sim \Phi_0 \frac{\Gamma(k)}{(2S_+ - 2S(Q))^k} \left(-\frac{1}{\pi} \cos \pi n/2\right) \exp \int_Q^{Q_+} dQ \frac{1}{\sqrt{2V(\cos Q)}} \quad (46)$$

5 The behaviour of the asymptotics in the vicinity of the minimum and high order behaviour of the ground state energy perturbation theory

The asymptotics obtained in the previous section for various cases have singularities in the origin. Namely, in one-dimensional case at small Q the asymptotic formula for Φ_k is approximately equal to

$$\Phi_k(Q) \sim \frac{B}{|Q|} \frac{(k-1)! k^\nu}{(A_0 - Q^2)^{k+\nu}}, \quad (47)$$

(the factor B depends on the potential), i.e. the pre-exponential factor diverges as $1/Q$. In the n -dimensional case there is a divergence as $1/|Q|^n$.

On the other hand, in the each order of semiclassical expansion the wave function is finite at zero. It is the behaviour of the eigenfunction near zero that allows us to find an asymptotic behaviour of the perturbation theory for eigenvalues.

It occurs, nevertheless, that the $\Phi_k(Q)$ asymptotics considered not under the conditions $Q = \text{const}, k \rightarrow \infty$ but under other conditions

$$k \rightarrow \infty, Q\sqrt{k} \rightarrow y = \text{const}, \quad (48)$$

is non-singular at zero argument. The knowledge of this asymptotics enables us to find high order behaviour of the eigenvalue perturbation theory.

It is convenient to change the variable $y = z\sqrt{A_0}$. Let us search for the Φ_k asymptotics under the conditions (48) in a form:

$$\frac{(k-1)!k^{\nu+1/2}}{A_0^{k+\nu+1/2}}g(z), \quad (49)$$

while the $E^{(k)}$ asymptotics is seeking in a form:

$$E^{(k)} \sim \frac{(k-1)!k^{\nu+1/2}}{A_0^{k+\nu+1/2}}\mathcal{E} \quad (50)$$

Consider the substitution of the formulas (49) and (50) to the equation (14). As the asymptotics is considered near the point $Q = 0$, one cannot omit the right-hand side of the equation (14). At larges k the main contribution to the sum in the right-hand side of the formula (14) is given by the k -th term equal to $E^{(k)}$ because $\Phi_0 = 1$ in the origin. Notice also that $\Delta S(0) = n$. Therefore, at larges k eq.(14) takes the following form in the leading order,

$$-\frac{1}{2}\frac{d^2g}{dz^2} + z\frac{dg}{dz} = \mathcal{E}. \quad (51)$$

Find now the connection between the quantity \mathcal{E} defining the large orders of the ground state energy perturbation theory and the factor B obtained from the wave function high order asymptotics.

Consider first the case of a symmetric potential. Formula (47) takes then place at positive Q as well as at negative Q . At larges z an asymptotic formula for $\Phi_k(z\sqrt{A_0/k})$ must transform to the asymptotics (47) where the following change, $Q = z\sqrt{A_0/k}$, is made. Let us substract the factor A_0^{-k} from eq.(47) and make a limit (48). Let us also take into account the relation, $(1 - z^2/k)^{k+\nu} \rightarrow e^{-z^2}$. The following boundary condition for function g at $+\infty$, as well as at $-\infty$ are obtained from it:

$$g(z) \sim \frac{B}{|z|} e^{z^2}. \quad (52)$$

On the other hand, eq.(51) can be solved exactly for the derivative $g'(z)$. Comparing this solution with the asymptotics for $g'(z)$ obtained from eq.(52), one can find that:

$$\mathcal{E} = -\frac{2B}{\sqrt{\pi}}.$$

The constant B and, therefore, the quantity \mathcal{E} can be found for various cases considered in the previous section. Thus, ground state energy perturbation theory has the following large order asymptotic behaviour,

$$E^{(k)} \simeq -\frac{k!k^{-1/2}}{\pi^{3/2}S_B^{k+1/2}}Q_+ \exp \int_0^{Q_+} dr [1/\sqrt{2V(r)} - 1/r],$$

in the case of ordinary turning point discussed in subsection 4.1, while in the case of the potential with degenerate minima treated in subsection 4.2 the asymptotics is as follows,

$$E^{(k)} \sim -\frac{2k!\sqrt{\omega}}{\pi A_0^{k+1}} \left(\frac{2k\omega}{A_0}\right)^{\frac{1-\omega}{2\omega}} \frac{Q_+^{1/\omega+1}}{\Gamma(\frac{1+\omega}{2\omega})} \\ \times \exp \int_0^{Q_+} dQ \left(\frac{1}{\sqrt{2V(Q)}} - \frac{1}{\omega(Q_+ - Q)} - \frac{1}{Q} \right).$$

These formulas are in agreement with the results obtained in [3, 4].

When the potential is not symmetric, the function A in the asymptotics (15) has discontinuity at zero argument. For the definiteness, assume that the least value of A corresponds to positive Q . Then the boundary condition at $+\infty$ has as in

the considered case the form (52), while the condition at $-\infty$ is the boundness of g . It follows from solving eq.(51) with these boundary conditions that the quantity \mathcal{E} is in 2 times lesser than in the previous case: $\mathcal{E} = -\frac{B}{\sqrt{\pi}}$. This result has the following interpretation in terms of the path integral approach. When the potential is even, there are two classical solutions symmetric under the change Q to $-Q$ which contribute to the asymptotics, as opposed to non-even potentials.

Consider now the case of Hamiltonians (27). The asymptotic formula for Φ_k transforms at small Q to the following expression,

$$\frac{(k-1)!}{(A_0 - Q^2)^k} \frac{B}{|Q|^n}.$$

Let us seek for the asymptotics of Φ_k under the conditions (47) in a form,

$$\frac{(k-1)!k^{n/2}}{A_0^{k+n/2}} g(Q\sqrt{k/A_0}), \quad (53)$$

analogous to one-dimensional case, where function $g(z)$ satisfies the following boundary conditions at larges z ,

$$g(z) \sim B e^{z^2}/z^n. \quad (54)$$

Let us look for the asymptotics of $E^{(k)}$ in a form,

$$E^{(k)} \sim \frac{(k-1)!k^{n/2}}{A_0^{k+n/2}} \mathcal{E}. \quad (55)$$

Substitution of the formulas (53) and (55) to the recursive relations of semiclassical expansion gives us the following equation for g ,

$$-\frac{1}{2}g''(z) - \frac{n-1}{2z}g'(z) + zg'(z) = \mathcal{E}.$$

Its solution which is an even function of z and equal to zero at $z = 0$ have the form

$$g(z) = \int_0^z \frac{dz}{z^{n-1}} e^{z^2} \int_0^z [-2\mathcal{E} z'^{n-1} e^{-z'^2}] dz'.$$

Asymptotic behaviour of this formula at large z is in agreement with the expression (54) when

$$\mathcal{E} = -\frac{2B}{\Gamma(n/2)}.$$

Substitution of the expression for B to this formula leads to the asymptotics (55) coinciding in the case of the $O(n)$ -symmetric systems with the results of [3],

$$E^{(k)} \simeq -\frac{k!k^{n/2-1}}{\pi\Gamma(n/2)S_B^{k+n/2}}(Q_+ \exp \int_0^{Q_+} dr [1/\sqrt{2V(r)} - 1/r])^n, \quad (56)$$

and to the asymptotics

$$E^{(k)} \simeq \frac{k!k^{n/2-1}}{S_{SI}^{k+n/2}} \cos(\pi n/2) \frac{2\pi^{n-1}}{\Gamma(n/2)} \exp \left(n \int_0^\pi d\theta \left(\frac{1}{\sqrt{2V(\cos \theta)}} - \frac{1}{\pi - \theta} \right) \right) \quad (57)$$

in the case of the Hamiltonians (42). The asymptotics (57) is in agreement with [6].

6 Conclusions

In this paper large order asymptotics of the tunnel semiclassical expansion for quantum mechanical systems is considered. As usual, the dependence of the Hamiltonian on the semiclassical expansion parameter (Planck constant) has the form (3). The results related to the high order asymptotics for the ground state energy are in agreement with the papers [11, 12, 3, 4, 5]. Contrary to them, this paper deals with the study of large order asymptotics not only for eigenvalues but also for eigenfunctions. Therefore, one can examine this problem not only by the path integral approach but also by the direct analysis of the semiclassical expansion recursive relations.

When one constructs the asymptotics, an important role is played by classical euclidean solutions starting from the origin and finishing at the point Q . These solutions give an exponentially small contribution to the wave function in comparison with the contribution of another solution satisfying the same boundary conditions. An interesting feature of the constructed asymptotics for the semiclassical expansion

large orders is the divergence of the pre-exponential factor near zero value of argument. This difficulty is resolved by the investigation of the asymptotics as $Q \sim 1/\sqrt{k}$. Another interesting feature of the obtained asymptotic formula is the following. In one-dimensional case the most essential growth of the asymptotics takes place near the turning point. It is the comparison of the asymptotics near this point with the exact coefficients of the expansion for the linear potential that allows us to find the unknown constants in the asymptotic formula. Analogous technique is applicable to other cases of the singular points.

Thus, the approach suggested in this paper enables us to obtain the high order asymptotics of semiclassical expansion both for eigenvalues and eigenfunctions. Although only the case of the ground state have been also examined, an analogous treatment can be also applicable to the excited states.

Probably, one can generalize the discussed method to quantum field theory and find the asymptotic behaviour of a sum of Feynman diagrams with N external lines and k loops as both N and k tend to infinity. The considered technique can be also useful in the case of instantons.

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Figure captions

Fig.1. Example of a one-dimensional potential $V(Q)$. This function has a local minimum as its argument equals to zero, $V(Q) \sim Q^2/2$, is positive when $0 < Q < Q_+$ and negative as $Q_+ < Q$.

Fig.2. Solid line: trajectory giving the main contribution to the ground state wave function. Dashed line: trajectory determining the large order behaviour of semiclassical expansion.

Fig.3. Potential $V(Q)$ with degenerate minima. This function has two minima as $Q = 0$ and $Q = Q_+$, $V(Q) = V(Q_+) = 0$, and is positive as $Q \neq 0, Q \neq Q_+$.

Fig.4. A particle moving on a two-dimensional sphere. Solid and dashed lines are, as well as in fig.2, classical euclidean solutions contributing to wave functions and to large order asymptotics.

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